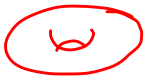
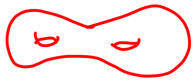
Pre-conference in Algebraic Geometry
MIST 2017/9.

§ Kodaira dimension

Classification X^n

$n=1$:			 ....
	$g=0$	$g=1$	$g\geqslant 2$
	$\mathbb{CP}^1$	$E_\tau := \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$	$\Sigma_g := D^2/\Gamma$
(Ricci) curv.	>0	$=0$	<0
$\parallel\!\mid$			
c_1	>0	$=0$	<0
	K_X^{-1} ample	$K_X \equiv \mathcal{O}_X$	K_X ample
Moduli sp:	1 pt.	$\tau \in$ 	$\mathcal{M}_g$ (Mumford G.I.T.)

$n>1$: Eg. $X^n = \{f=0\} \subseteq \mathbb{CP}^{n+1}$ <u>hypersurface.</u>

$K_X^{-1} = \mathcal{O}_X(n+2-d)$ where $d = \deg X := \deg f$

So $c_1(X) > 0 \iff d \leqslant n+1$ and so on.

(reason: $\deg f = d \Rightarrow N_{X/\mathbb{P}^{n+1}} = \mathcal{O}(d)$
Euler seq. $\Rightarrow K^{-1}_{\mathbb{P}^{n+1}} = \mathcal{O}(n+2)$ $\Big\} \Rightarrow K_X^{-1} = \mathcal{O}(n+2-d)$

- $K_X \;<\; = \;>\; 0$ is too loose in general.

Kodaira dimension:

$\kappa(X^n) = \quad -\infty \quad 0 \quad 1 \quad 2 \quad \cdots\cdots \quad n$

$\qquad\qquad\quad \cup\!\!\mid \qquad\qquad\qquad\qquad\qquad\qquad \cup\!\!\mid$

(eg. $\quad c_1>0 \quad c_1=0 \qquad\qquad\qquad c_1<0$)

($\kappa(X)$ "=" # direct$^{\underline{s}}$. $K_X > 0$, other direct. $= 0$.)

($\kappa(\mathbb{P}^1 \times Y^{n-1}) = -\infty$; $\kappa((\Sigma_g)^{\ell} \times (T_\tau)^{n-\ell}) = \ell$)

Recall: Riemann-Roch formula.

$\mathbb{C} \to L \to X$, $\sum_{i=0}^{n}(-1)^i \dim H^i(X,L) = \int_X ch(L)\, Td(T_X)$

$$ch(L) = e^{c_1(L)} = 1 + c_1(L) + \frac{c_1^2(L)}{2} + \dots \in H^2(X,\mathbb{Q})$$

$$Td(T_X) = 1 + \frac{c_1(X)}{2} + \frac{c_1^2(X)+c_2(X)}{12} + \frac{c_1(X)c_2(X)}{24} + \dots$$

Riemann-Roch formula:

$$\sum_{i=0}^{n}(-1)^i \dim H^0(X, K_X^{\otimes k}) = \int_X \overbrace{ch(K_X^{\otimes k})}^{e^{-k c_1(X)}}\ \overbrace{Td(X)}^{1+\text{h.o.t.}}$$

$\|$ if $K_X > 0$, $k \gg 0$ $\qquad\qquad$ $\|$

$\dim H^0(X, K_X^{\otimes k})$ $\qquad\qquad$ $\left(\int \frac{-c_1(X)^n}{n!}\right) \cdot k^n + O(k^{n-1})$

$\vee$

0 if $K_X > 0$

For general X, consider $k \gg 0$

$$P_k(X) \triangleq \dim H^0(X, K_X^{\otimes k}) = c\, k^d + O(k^{d-1}) \quad \exists\, d$$

Call $d = \kappa(X)$ Kodaira dimension of X

($P_k(X)$'s are pluri-canonical genus)

Write $\kappa(X) = -\infty$ if $H^0(K_X^{\otimes k}) = 0$ $\forall\, k \gg 0$.

Remark: $\kappa(X) = \dim\left(\Phi_{|kK_X|}(X)\right)$ for $k \gg 0$

where $\Phi_{|kK_X|} : X \dashrightarrow \mathbb{P}(H^0(K_X^{\otimes k}))$ pluri-canonical map.

Recall: $\Phi_{|L|} : X \dashrightarrow \mathbb{P}(H^0(X,L)) \simeq \mathbb{CP}^N$.

$\qquad x \longmapsto [s_0(x), \dots, s_N(x)]$

if $s_0, \dots, s_N$ is a base for $H^0(X,L) \cong \mathbb{C}^N$.

§ Iitaka conjecture

$$X \twoheadrightarrow Y \overset{?}{\Longrightarrow} \kappa(X) \geq \kappa(Y) + \kappa\begin{pmatrix}\text{generic}\\ \text{fiber}\end{pmatrix}$$

X, Y smooth

True if $\dim X = 2$ (see Barth-Peters-Van de Ven)

(Jungkai Chen. Effective Iitaka fibration of 3folds)

§ General type manifolds. (i.e. $\kappa(X) = \dim X$)

($\sim$ $c_1(X) \leq 0$, say nef & big)

Eg. K_X ample $\Longleftrightarrow$ $c_1(X) < 0$ $\Longleftrightarrow$ $Ric(\omega_X) < 0$

$\overset{\text{Aubin, Yau}}{\Longleftrightarrow}$ $Ric(\omega_X) = -\omega_X$ $\exists!\ \omega_X$ i.e. Kähler-Einstein

$\Rightarrow$ Chern# ineqt. $(-1)^n c_1^n \leq (-1)^n\, 2\frac{n+1}{n}\, c_1^{n-2} c_2$

"=" $\Leftrightarrow$ bisect $= -1$ $\Longleftrightarrow$ $\tilde{X} = B^n_{\mathbb{C}}$ (uniformization)

$[n=1 \Rightarrow \Sigma_{g\geq 2}]$ $\exists$ moduli space $\mathcal{M}_g$ (of $\dim_{\mathbb{C}}$ $3g-3$)

$$\mathcal{M}_g = \mathcal{T}_g / \Gamma$$ (use Mumford G.I.T.)

$\mathcal{T}_g$: Teichmüler sp. (cpx. str., up to diffeo. fixing $H^1(\Sigma_g, \mathbb{Z})$ (marking))

Γ: mapping class group $\Gamma = Aut(\mathbb{Z}^{2g}, \cup)$

$$\mathcal{T}_g \underset{\text{bdd domain}}{\subset} \mathbb{C}^{3g-3} \Longrightarrow \text{curv.} \leq 0$$

$\rightsquigarrow$ $X^2 \xrightarrow{f} \Sigma^1$ $\Longrightarrow$ $\deg(R^1 f_* \mathcal{O}_X) \leq 0$

relative min.

"$\Sigma' \to \overline{\mathcal{M}}_g$"

$\cdots \Rightarrow$ Iitaka conj. when $\dim X = 2$.

[$n=2$] $\exists$ moduli space
(using Minimal Model Program (MMP) for 3-folds)
$\wr$
"Classification problem"

(constr. moduli ~ 1 parameter family of surfaces $\rightsquigarrow$ 3 fold)

Inevitably, we need to allow some "mild singularities", e.g. <u>terminal</u> (smooth if $n=2$), <u>canonical</u> ($\mathbb{C}^2/\Gamma$, $\Gamma \leq SU(2)$ if $n=2$).
Also, really need $K_X^{\otimes k}$ exists, i.e. <u>$\mathbb{Q}$-factorial</u>.

(Kollár. Moduli of varieties of general type.)

§ Fano manifolds. (in particular, $\kappa = -\infty$)
$c_1(X) > 0$ [$\dim X = 1 \Rightarrow X = \mathbb{P}^1$]

[$\dim X = 2$] X = blowup $\mathbb{P}^2$ at $n \leq 8$ points in general positions, or $\mathbb{P}^1 \times \mathbb{P}^1$

Also called del Pezzo surface.
($\sim$ exceptional Lie group E_n)

- $n \leq 6 \Rightarrow K^{-1}$ very ample (i.e. $X \xhookrightarrow{\Phi_{K^{-1}}} \mathbb{P}^{9-n}$)
  $n = 7$ or $8 \Rightarrow K^{-2}$ very ample ?

[$\dim X = 3$] Fano 3-folds have been completely classified.

(Meng Chen. On the anti-canon. geometry of weak $\mathbb{Q}$-Fano 3-folds.)

"weak Fano" means $c_1(X)$ <u>nef</u> & <u>big</u>. ($\sim c_1 \geq 0$ & $c_1^n > 0$)

$\forall$ curve, $\int_C c_1(X) \geq 0$ $\int c_1(X)^n > 0$

§ Calabi-Yau manifolds. (In particular $\kappa = 0$)

i.e. $K_X \simeq \mathcal{O}_X$ ($\sim$ $c_1(X) = 0$)

(Yau) $\exists!\, \omega_X$ in every Kähler class,

s.t. $\mathrm{Ricci}(\omega_X) = 0$

- Mirror Symmetry Conjecture.

[$n=1$] $X = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ elliptic curve ω

[$n=2$] $X = \mathbb{C}^2/\Lambda$ Complex torus/Abelian variety

or K3 surface ($c_1 = 0 = b_1$)

(Voisin, Segre classes of tautological bundles on Hilbert schemes of surfaces).

$$c(E) = \prod_i (1 + \underbrace{x_i}_{\text{Chern root}}) \Rightarrow \text{Segre class } s(E) = \prod_i \frac{1}{1+x_i}$$

$$c_1 = -s_1, \quad c_2 = s_1^2 - s_2, \cdots, c_n = -s_1 c_{n-1} - s_2 c_{n-2} - \cdots - s_n$$

$$\mathbb{C} \to L \to X \quad \text{surface},$$

$$\rightsquigarrow \mathbb{C}^k \to L_{[k]} \to S^{[k]}X$$ ← Hilbert scheme of k points on X.

Thm: X K3 surface, $c_1(L)^2 =: 2g-2$

$$\Rightarrow s_{k,g} \triangleq \int_{S^{[k]}X} \underbrace{s_{\mathrm{TOP}}}_{2k}(L_{[k]}) = 2^k \binom{g-2k+1}{k}$$

Cor: Under same assumptions

$$s_{k,g} = 0 \quad \text{if} \quad k > g - 2k + 1 \geq 0$$

Remark: $\dim X = 1 \Rightarrow S^{[k]}X = (\prod^k X)/S_k =: S^k X$

$\dim X = 2 \Rightarrow S^{[k]}X \twoheadrightarrow S^k X$ is crepant resol$^{\underline{n}}$.

in particular smooth.

e.g. $S^{[2]}X = \mathrm{Blow}_{\Delta_X} S^2 X$

$L_{[k]}$ over $(x_1, \dots, x_k) \in S^{[k]}X \underset{\text{if } x_i\text{'s distinct}}{=\!=} \bigoplus_i^k L_{x_i}$

Remark: For any surface X & $\forall$ L

$\int_{S^{[k]}X} s_{2k}(L_{[k]})$ depends only on topology,

(i.e. $c_1^2(X)$, $c_2(X)$, $c_1^2(L)$, $c_1(L)\cdot c_1(X)$).

($\rightsquigarrow$ Cor $\Rightarrow$ Thm.)